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Scattering of scalar and Dirac particles by a magnetic tube of finite radius

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Abstract. We consider the Dirac equation in cylindrically symmetric magnetic fields and find its normal modes as eigenfunctions of a complete set of commuting operators. This set consists of the Dirac operator itself, the z-components of the linear and the total angular momenta, and of one of the possible spin polarization operators. The spin structure of the solution is completely fixed independently of the radial distribution of the magnetic field which influences only the radial modes.

We solve explicitly the radial equations for the uniform magnetic field inside a solenoid of finite radius and consider in detail the scattering of scalar and Dirac particles in this field. For particles with low energy the scattering cross section coincides with the Aharonov–Bohm scattering cross section. We work out the first-order corrections to this result caused by the fact that the solenoid radius is finite. At high energies we obtain the classical result for the scattering cross section.

1. Introduction

The behaviour of relativistic charged particles in external magnetic fields has been the subject of many investigations in QED (see, for example, [2, 6]), most of which have been concerned with the synchrotron radiation emitted by charged particles moving in cyclic accelerators and storage rings [16]. This effect has found wide applications in different fields of physics, biology and technology.

By its origin the synchrotron radiation is the classical effect. It has been shown that quantum corrections became important only for relativistic particles or intense magnetic fields [10, 16]; this happens at high energies E and high field strengths B leading to a characteristic product [14]

$$\frac{B}{B_0}\frac{E}{Mc^2} \approx 1 \qquad \text{with } B_0 = \frac{M^2c^3}{e\hbar} \tag{1}$$

of the order 1. Other QED processes, such as pair production and pair annihilation, have no classical counterparts. In these cases, intensities of the processes also become relevant if particle energies and magnetic fields fulfil (1). Therefore, in QED processes, the external magnetic field cannot be treated as a perturbation. The relativistic particles must instead be described by exact solutions of the Dirac equation in an external magnetic field; this will

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also be done in this paper. For the comprehensive review of exact solutions of relativistic wave equations in external fields see [5].

Much work has been devoted to solutions of the Dirac equation in uniform electric and magnetic fields [7, 13] and in the electromagnetic plane wave field [17]. Different QED processes have been studied for the corresponding physical situations [7, 13].

A new field of research was initiated with the study of the Aharonov–Bohm (AB) effect (for the latter see [1, 12]). QED processes in the presence of a magnetic string have been elaborated in detail. The differential cross sections of the bremsstrahlung of an electron passing by the magnetic string [4] and pair production by a single photon in this potential [15] have been worked out. In these cases it surprisingly turned out that the cross sections do not become small even for low particle energies. This fact can be interpreted as follows: for a finite flux the string magnetic field becomes infinitely strong in the limit of a string, so that the criterion of equation (1) is formally fulfilled for all energies.

To describe realistic situations the magnetic string has to be replaced by a thin solenoid which contains an intense but finite magnetic field. It is now very important to know how the results obtained in the string limiting case are modified for a solenoid of finite radius. We address this problem, in this and following papers, and investigate different QED processes in cylindrical magnetic fields in detail. Below we concentrate on the scattering of scalar and Dirac particles.

In sections 2 and 3 we define the normal modes of the Dirac equation in cylindrical magnetic fields of arbitrary radial dependence as eigenfunctions of a complete set of commuting operators. The set consists of the Dirac operator itself, the z-components of the linear and total angular momenta, and also of one of the possible spin polarization operators. We fix completely the spin structure of the solution independently of the radial distribution of the magnetic field, which influences only the radial modes. In section 4 we solve explicitly the radial equations for the model of a solenoid of finite radius with a uniform magnetic field. With the tube radius tending to zero, this model can be considered as a realistic model replacing the AB magnetic string. This approach allows us to solve correctly the Dirac equation for the AB potential. Then we consider in detail (section 5) the scattering of scalar and Dirac particles by the magnetic field of the solenoid of finite radius. At low energies of incident particles the scattering cross section coincides with the AB scattering cross section. We derive the first-order corrections caused by the finite tube radius. At high energies we obtain the classical result for the scattering cross section.

Throughout we use units such that $\hbar = c = 1$.

2. The Dirac equation in a cylindrically symmetric magnetic field

The Dirac equation in an external magnetic field reads in cylindrical coordinates (ρ, φ, z)

$$\mathrm{i}\partial_t \Psi(x) = H\Psi(x) \qquad H = \alpha_i(\hat{p}_i - eA_i) + \beta M \tag{2}$$

where $x = (t, \rho, \varphi, z)$, e is the charge of the Dirac particle ($e > 0$ for the positron and $e < 0$ for the electron). α_i and β are the known matrices written in cylindrical coordinates, and the kinetic momenta are given by

$$\hat{\pi}_\rho := \hat{p}_\rho = -\mathrm{i}\partial_\rho \qquad \hat{p}_3 := -\mathrm{i}\partial_z$$
$$\hat{\pi}_\varphi := \hat{p}_\varphi - eA_\varphi = -\frac{\mathrm{i}}{\rho}\partial_\varphi - eA_\varphi. \tag{3}$$

The vector potential for an *arbitrary* cylindrically symmetric magnetic field of a fixed direction (along the z-axis) has a non-zero angular component $A_\varphi(\rho)$, and the magnetic field reads in terms of the potential

$$B_z(\rho) = A'_\varphi(\rho) + \frac{1}{\rho} A_\varphi(\rho). \tag{4}$$

The dependence of the solutions on z and the azimuthal angle φ can be fixed by demanding that $\Psi(x)$ is an eigenfunction of the operators of the linear momentum projection $\hat{p}_3$ and total angular momentum projection $\hat{J}_3$,

$$\begin{aligned} &\hat{p}_3\Psi(x) = p_3\Psi(x) \\ &\hat{J}_3\Psi(x) = (\hat{L}_3 + \tfrac{1}{2}\Sigma_3)\Psi(x) = (-\mathrm{i}\partial_\varphi + \tfrac{1}{2}\Sigma_3)\Psi(x) = j_3\Psi(x) \qquad j_3 := l + \tfrac{1}{2}. \end{aligned} \tag{5}$$

Here l is the integral part of the half-integral eigenvalue j_3.

We find

$$\Psi(x) = \frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}E_p t + \mathrm{i}p_3 z}\psi(\rho,\varphi) \qquad \psi(\rho,\varphi) = \begin{pmatrix} u \\ v \end{pmatrix} \tag{6}$$

where

$$u = \begin{pmatrix} C_1 R_1(\rho)\,\mathrm{e}^{\mathrm{i}l\varphi} \\ C_2 R_2(\rho)\,\mathrm{e}^{\mathrm{i}(l+1)\varphi} \end{pmatrix} \qquad v = \begin{pmatrix} C_3 R_3(\rho)\,\mathrm{e}^{\mathrm{i}l\varphi} \\ C_4 R_4(\rho)\,\mathrm{e}^{\mathrm{i}(l+1)\varphi} \end{pmatrix} \tag{7}$$

$E_p = \sqrt{p^2 + M^2} = \sqrt{p_\perp^2 + p_3^2 + M^2}$ is the energy, $p_\perp$ denotes the radial momentum and C_i are the spin polarization coefficients.

In terms of u- and v-spinors the Dirac equation reads

$$\begin{aligned} \sigma_i(p_i - eA_i)u &= (E_p + M)v \\ \sigma_i(p_i - eA_i)v &= (E_p - M)u. \end{aligned} \tag{8}$$

From equations (8) and (7) we find equations for the radial functions R_i,

$$\begin{aligned} &-\mathrm{i}C_2R_2' - \mathrm{i}\left(\frac{l+1}{\rho} - eA_\varphi\right)C_2R_2 + p_3C_1R_1 = (E_p + M)C_3R_3 \\ &-\mathrm{i}C_1R_1' + \mathrm{i}\left(\frac{l}{\rho} - eA_\varphi\right)C_1R_1 - p_3C_2R_2 = (E_p + M)C_4R_4 \\ &-\mathrm{i}C_4R_4' - \mathrm{i}\left(\frac{l+1}{\rho} - eA_\varphi\right)C_4R_4 + p_3C_3R_3 = (E_p - M)C_1R_1 \\ &-\mathrm{i}C_3R_3' + \mathrm{i}\left(\frac{l}{\rho} - eA_\varphi\right)C_3R_3 - p_3C_4R_4 = (E_p - M)C_2R_2. \end{aligned} \tag{9}$$

The spin coefficients can be constrained by the normalization condition

$$\sum_{i=1}^{i=4} |C_i|^2 = 1. \tag{10}$$

3. Spin polarization states

In the case of cylindrical magnetic fields the spin polarization coefficients C_i can be fixed by one of two spin operators $\hat{S}_t$ or $\hat{S}_3$ [16] which commute with operators $\hat{H}$, $\hat{p}_3$ and $\hat{J}_3$ but do not commute with each other. The helicity operator,

$$\hat{S}_t := \frac{\boldsymbol{\Sigma}\cdot(\hat{\boldsymbol{p}} - e\boldsymbol{A})}{p} \tag{11}$$

describes longitudinal polarization (the projection of the spin onto the velocity direction of the Dirac particle). The z-component S_3 of the operator $\hat{\boldsymbol{S}} := \beta\boldsymbol{\Sigma} + \gamma(\hat{\boldsymbol{p}} - e\boldsymbol{A})/M$,

$$\hat{S}_3 := \beta\Sigma_3 + \gamma\frac{\hat{p}_3}{M} \qquad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

defines a transverse polarization (along the direction of the magnetic field) for non-relativistic motion, or for the motion in the plane perpendicular to the magnetic field. The operators $\hat{S}_t$ and $\hat{S}_3$ are not independent, but to make things easy for the reader we will discuss them separately.

3.1. The helicity states

For the solution of the Dirac equation (2) which is the eigenstate of the helicity operator (11),

$$\hat{S}_t\Psi(x) = s\Psi(x) \qquad s = \pm 1 \tag{13}$$

equations (8) and (11) connect the u- and v-spinors according to

$$v = s\frac{\sqrt{E_p - M}}{\sqrt{E_p + M}}u. \tag{14}$$

This means that $R_3 = R_1, R_4 = R_2$ and that the coefficients C_3, C_4 are coupled by equation (14) to C_1, C_2 correspondingly. Choosing C_1 and C_2 to be

$$C_1 = \frac{\sqrt{E_p + M}\sqrt{p + sp_3}}{\sqrt{2E_p}\sqrt{2p}} \qquad C_2 = \frac{\mathrm{i}s\sqrt{E_p + M}\sqrt{p - sp_3}}{\sqrt{2E_p}\sqrt{2p}} \tag{15}$$

and determining coefficients C_3 and C_4 from (14),

$$C_3 = \frac{s\sqrt{E_p - M}\sqrt{p + sp_3}}{\sqrt{2E_p}\sqrt{2p}} \qquad C_4 = \frac{\mathrm{i}\sqrt{E_p - M}\sqrt{p - sp_3}}{\sqrt{2E_p}\sqrt{2p}} \tag{16}$$

we obtain from (9) the following equations for the independent radial components $R_{1,2}$,

$$R_2' + \left(\frac{l+1}{\rho} - eA_\varphi\right)R_2 = p_\perp R_1 \qquad - R_1' + \left(\frac{l}{\rho} - eA_\varphi\right)R_1 = p_\perp R_2. \tag{17}$$

The solution of the Dirac equation with quantum numbers E, p_3, j_3, s then has the form of (6) and (7) with $R_3 = R_1$, $R_4 = R_2$ and coefficients (15) and (16).

3.2. The transverse polarization states

Alternatively, for the eigenstate of the operator $\hat{S}_3$ (12),

$$\hat{S}_3\Psi(x) = s_3\Psi(x) \qquad s_3 = s\sqrt{1 + \frac{p_3^2}{M^2}} \qquad s = \pm 1 \tag{18}$$

we find from (8) and (12) that

$$v_1 = \frac{p_3}{M(s_3 + 1)}u_1 \qquad v_2 = \frac{p_3}{M(s_3 - 1)}u_2. \tag{19}$$

This means that $R_3 = R_1, R_4 = R_2$ and that the coefficients C_3, C_4 are correspondingly defined by (19) from C_1, C_2. Fixing the coefficients C_1 and C_2 by the expressions

$$C_1 = \sqrt{\frac{s_3 + 1}{2s_3}}\frac{\sqrt{E_p + s_3M}}{\sqrt{2E_p}} \qquad C_2 = \mathrm{i}\epsilon(s_3p_3)\sqrt{\frac{s_3 - 1}{2s_3}}\frac{\sqrt{E_p - s_3M}}{\sqrt{2E_p}} \tag{20}$$

where $\epsilon(x)$ is the sign of x and determining the coefficients the C_3 and C_4 from (19),

$$C_3 = \epsilon(s_3 p_3)\sqrt{\frac{s_3 - 1}{2s_3}}\frac{\sqrt{E_p + s_3 M}}{\sqrt{2E_p}} \qquad C_4 = \mathrm{i}\sqrt{\frac{s_3 + 1}{2s_3}}\frac{\sqrt{E_p - s_3 M}}{\sqrt{2E_p}} \tag{21}$$

we again obtain from (9) the radial equations (17). The solution of the Dirac equation with the quantum numbers E, p_3, j_3, s_3 has, therefore, the form (6) and (7) with $R_3 = R_1$, $R_4 = R_2$ and the coefficients (20) and (21).

We have shown how the spin polarization operators (11) or (12) fix the spin structure of the solutions of the Dirac equation in an arbitrary cylindrical magnetic field. Note that solutions for different configurations of the magnetic field differ from each other only by their radial functions.

3.3. *The charge conjugate states*

The solutions (6) of the Dirac equation (2) represent electron ($e < 0$) and positron ($e > 0$) wavefunctions of positive energy, $\Psi_{\mathrm{e,p}}(x)$. The complete set of solutions of the Dirac equation (2) includes the positive and negative energy electron (or positron) states. Instead of negative energy electron (or positron) states one can also use the charge conjugated positron (electron) states which can be obtained by the charge conjugation operation,

$$\Psi(x) \to \Psi_{\mathrm{c}}(x) := C\overline{\Psi}_{\mathrm{transp}}(x) \qquad C = \alpha_2. \tag{22}$$

For the helicity and transverse states we obtain

$$\Psi_{\mathrm{c}}(x) = \frac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}E_p t - \mathrm{i}p_3 z}\psi_{\mathrm{c}}(\rho,\varphi), \psi_{\mathrm{c}}(\rho,\varphi) = \begin{pmatrix} u_{\mathrm{c}} \\ v_{\mathrm{c}} \end{pmatrix} \tag{23}$$

where

$$u_{\mathrm{c}} = \begin{pmatrix} \mathrm{i}C_4^* R_2^*(\rho)\,\mathrm{e}^{-\mathrm{i}(l+1)\varphi} \\ -\mathrm{i}C_3^* R_1^*(\rho)\,\mathrm{e}^{-\mathrm{i}l\varphi} \end{pmatrix} \qquad v_{\mathrm{c}} = \begin{pmatrix} -\mathrm{i}C_2^* R_2^*(\rho)\,\mathrm{e}^{-\mathrm{i}(l+1)\varphi} \\ \mathrm{i}C_1^* R_1^*(\rho)\,\mathrm{e}^{-\mathrm{i}l\varphi} \end{pmatrix} \tag{24}$$

with the coefficients (15), (16) and (20) and (21), correspondingly.

With reference to these states, the electron–positron field operator reads

$$\hat{\psi}(x) = \sum_j [\Psi_{\mathrm{e}}(x) a_j + \Psi_{\mathrm{p}}^c(x) b_j^\dagger] \tag{25}$$

where a_j and b_j are the annihilation operators for electrons and positrons with quantum numbers $j := (p_\perp, p_3, l, s)$. This field operator obeys equation (2) with $e < 0$; of course, we could use the charged conjugated field operator which obeys this equation with $e > 0$.

3.4. *The radial equations*

The set of radial equations (17) is equivalent to the following second-order equation for the first component

$$R_1'' + \frac{1}{\rho}R_1' - \left[\left(\frac{l}{\rho} - eA_\varphi\right)^2 - e\left(A_\varphi' + \frac{1}{\rho}A_\varphi\right) - p_\perp^2\right]R_1 = 0. \tag{26}$$

The second component is then defined by the equation

$$R_2 = \frac{1}{p_\perp}\left[-R_1' + \left(\frac{l}{\rho} - eA_\varphi\right)R_1\right]. \tag{27}$$

Alternatively, one could start with the second-order equation for R_2 and define the first component R_1 by equation (17). Both ways are completely equivalent. We would like to stress that because of the first-order constraints (17) between the components R_1 and R_2, the Dirac equation differs from the Klein–Gordon equation for relativistic scalar particles.

The radial functions can be normalized by the condition

$$\int \rho \, \mathrm{d}\rho R_{1,2}^{*}(p'_{\perp}\rho) R_{1,2}(p_{\perp}\rho) = \frac{\delta(p_{\perp} - p'_{\perp})}{\sqrt{p_{\perp} p'_{\perp}}}. \tag{28}$$

Equations (26) and (27) together with the normalization condition (28) define the radial functions R_1 and R_2 up to a common phase factor.

The non-relativistic limit. The positron and electron radial and angular functions conserve their forms in the non-relativistic approximation ($E_p = M + \mathcal{E}_p, \mathcal{E}_p \ll M$) with non-zero spin coefficients

$$C_1 = \frac{\sqrt{p + sp_3}}{\sqrt{2p}} \qquad C_2 = \frac{\sqrt{p - sp_3}}{\sqrt{2p}} \qquad \text{for helicity states}$$

$$C_1 = \frac{\sqrt{s+1}}{\sqrt{2s}} \qquad C_2 = \frac{\sqrt{s-1}}{\sqrt{2s}} \qquad \text{for transverse states.} \tag{29}$$

4. Solenoid of finite radius and the magnetic string

Our basic aim is to study the AB scattering of scalar and Dirac particles for experimentally realistic situations. We consider, therefore, a solenoid of finite radius ρ_0 with the uniform magnetic field inside. This model contains the AB magnetic string in the limit of zero radius and constant magnetic flux. This limit can alternatively be based on a different model [9] for which the magnetic field is concentrated on the surface of a cylinder of radius ρ_0. It is obvious that this second model being more simple for the investigation of zero radius limit does not describe a realistic experimental set-up.

4.1. The uniform magnetic field inside a solenoid of finite radius

The uniform magnetic field $(0, 0, B)$ restricted to the interior of a solenoid of finite radius ρ_0 can be described by the vector potential

$$eA_{\varphi} = \begin{cases} eB\rho/2 \\ e\Phi/2\pi\rho \end{cases} = \begin{cases} \phi\rho/\rho_0^2 & \text{if } \rho \leqslant \rho_0 \\ \phi/\rho & \text{if } \rho \geqslant \rho_0 \end{cases} \tag{30}$$

where $\Phi = \pi\rho_0^2 B := \beta\Phi_0 = \varepsilon\phi\Phi_0$ is the magnetic flux in the solenoid, $\Phi_0 := 2\pi/|e|$ is the magnetic flux quantum and ε is the sign of the charge ($\varepsilon > 0$ for positrons and $\varepsilon < 0$ for electrons). We take $\beta > 0$ so that the constant magnetic field inside the solenoid

$$B = \frac{2\beta}{|e|\rho_0^2} \qquad \rho \leqslant \rho_0 \tag{31}$$

points in the positive z direction. To obtain the uniform magnetic field in the whole space, one may perform the limiting procedure $\beta \to |e|B\rho_0^2/2$ together with $\rho_0 \to \infty$.

Let us consider first the internal solution. The general radial equations (26) and (27) read in this case

$$R_1'' + \frac{1}{\rho} R_1' - \left[\left(\frac{l}{\rho} - \frac{\phi}{\rho_0^2}\rho \right)^2 - 2\frac{\phi}{\rho_0^2} - p_{\perp}^2 \right] R_1 = 0 \tag{32}$$

$$R_2 = \frac{1}{p_{\perp}} \left[-R_1' + \left(\frac{l}{\rho} - \frac{\phi}{\rho_0^2}\rho \right) R_1 \right]. \tag{33}$$

Introducing the new variable $x = \beta(\rho^2/\rho_0^2)$ and the new functions $Y_{1,2}(x)$ according to

$$R_1(x) = x^{\frac{1}{2}|l|}\,\mathrm{e}^{-\frac{1}{2}x}\,Y_1(x) \qquad R_2(x) = x^{\frac{1}{2}|l+1|}\,\mathrm{e}^{-\frac{1}{2}x}\,Y_2(x) \tag{34}$$

we obtain from (32) the confluent hypergeometric equation [8]

$$\begin{aligned} & xY_1'' + (B_1 - x)Y_1' - A_1Y_1 = 0 \\ & A_1 = \frac{|l| - \varepsilon l + 1 - \varepsilon}{2} - \frac{p_\perp^2\rho_0^2}{4\beta} \qquad B_1 = |l| + 1. \end{aligned} \tag{35}$$

Then from equations (33)–(35) we find the following confluent hypergeometric equation for the function $Y_2(x)$

$$\begin{aligned} & xY_2'' + (B_2 - x)Y_2' - A_2Y_2 = 0 \\ & A_2 = \frac{|l+1| - \varepsilon(l+1) + 1 + \varepsilon}{2} - \frac{p_\perp^2\rho_0^2}{4\beta} \qquad B_2 = |l+1| + 1. \end{aligned} \tag{36}$$

Solutions of equations (35) and (36) which are regular at $\rho = 0$ are the confluent hypergeometric functions $\Phi(A_1, B_1; x)$ and $\Phi(A_2, B_2; x)$, correspondingly, so that we obtain

$$\begin{aligned} & R_1^{\mathrm{int}}(x) = c_l x^{\frac{1}{2}|l|}\,\mathrm{e}^{-\frac{1}{2}x}\,\Phi(A_1, B_1; x) \\ & R_2^{\mathrm{int}}(x) = \tilde{c}_l x^{\frac{1}{2}|l+1|}\,\mathrm{e}^{-\frac{1}{2}x}\,\Phi(A_2, B_2; x). \end{aligned} \tag{37}$$

The coefficients c_l and $\tilde{c}_l$ are connected through (33).

The external solution obeys the radial equations (26) and (33) at $\rho > \rho_0$:

$$\begin{aligned} & R_1'' + \frac{1}{\rho}R_1' - \frac{(l-\phi)^2}{\rho^2}R_1 + p_\perp^2 R_1 = 0 \\ & R_2 = \frac{1}{p_\perp}\left(-R_1' + \frac{l-\phi}{\rho}R_1\right). \end{aligned} \tag{38}$$

Solving these equations by Bessel functions of the first kind with positive and negative order and using (27) we find the external radial components

$$\begin{aligned} & R_1^{\mathrm{ext}}(\rho) = a_l J_\nu(p_\perp\rho) + b_l J_{-\nu}(p_\perp\rho) \\ & R_2^{\mathrm{ext}}(\rho) = \epsilon_{l-\phi}[a_l J_{\tilde{\nu}}(p_\perp\rho) - b_l J_{-\tilde{\nu}}(p_\perp\rho)] \end{aligned} \tag{39}$$

where

$$\nu := |l - \phi| \qquad \tilde{\nu} := \nu + \epsilon_{l-\phi} \qquad \epsilon_{l-\phi} := \begin{cases} 1 & \text{if } l > \phi \\ -1 & \text{if } l < \phi. \end{cases} \tag{40}$$

The coefficients a_l and b_l can be obtained from matching conditions at the surface $\rho = \rho_0$ or ($x = \beta$) of the solenoid for internal and external solutions. Since radial components are connected through (33), one can use any couple of the matching conditions for components R_1, R_2 and their first derivatives R_1', R_2'. Choosing the conditions for R_1 and R_1' we obtain

$$\begin{aligned} & c_l\beta^{\frac{1}{2}|l|}\,\mathrm{e}^{-\frac{1}{2}\beta}\,\Phi(A_1, B_1; \beta) = a_l J_\nu(p_\perp\rho_0) + b_l J_{-\nu}(p_\perp\rho_0) \\ & \frac{2\beta}{p_\perp\rho_0}\left[\frac{|l| - \beta}{2\beta} + \frac{\Phi'(A_1, B_1; \beta)}{\Phi(A_1, B_1; \beta)}\right] c_l\beta^{\frac{1}{2}|l|}\,\mathrm{e}^{-\frac{1}{2}\beta}\,\Phi(A_1, B_1; \beta) \\ & \qquad = a_l J_\nu'(p_\perp\rho_0) + b_l J_{-\nu}'(p_\perp\rho_0). \end{aligned} \tag{41}$$

Eliminating the coefficient c_l from these equations we find the matching conditions in the form

$$\frac{2\beta}{p_\perp\rho_0}\left[\frac{|l|-\beta}{2\beta}+\frac{\Phi'(A_1,B_1;\beta)}{\Phi(A_1,B_1;\beta)}\right]=\frac{2\beta}{p_\perp\rho_0}\left[\frac{|l|-\beta}{2\beta}+\frac{A_1}{B_1}\frac{\Phi(A_1+1,B_1+1;\beta)}{\Phi(A_1,B_1;\beta)}\right]$$
$$=\frac{a_l J'_\nu(p_\perp\rho_0)+b_l J'_{-\nu}(p_\perp\rho_0)}{a_l J_\nu(p_\perp\rho_0)+b_l J_{-\nu}(p_\perp\rho_0)} \tag{42}$$

which fixes all coefficients up to a normalization constant. The same results can be obtained from the continuity conditions for R_1 and R_2 at $\rho=\rho_0$, for instance.

4.2. The zero radius limit (magnetic string)

The vector potential for the infinitely thin, infinitely long straight magnetic string lying along the z-axis

$$eA_\varphi=\frac{e\Phi}{2\pi\rho}=\frac{\varepsilon\Phi}{\Phi_0\rho}=\frac{\varepsilon\beta}{\rho}=\frac{\phi}{\rho}\qquad 0\leqslant\rho<\infty \tag{43}$$

can be obtained from the vector potential of the solenoid (30) for $\rho_0\to 0$ keeping the magnetic flux constant. It is singular on the z-axis and produces the singular magnetic field

$$B=\frac{2\beta}{|e|\rho}\delta(\rho) \tag{44}$$

which is concentrated on the z-axis and points in the positive z direction. We separate the integral number N from the flux parameter β, $\beta=N+\delta$, $N\geqslant 0, 0\leqslant\delta<1$, since it is the fractional part δ of the dimensionless magnetic flux which produces all physical effects. Its integral part N will appear as a phase factor $\exp(\mathrm{i}\varepsilon N\varphi)$ in the solutions to the Dirac equation.

In the case of the AB potential the radial solutions are given by (39), but have another domain of definition. We note that the component R_1 contains modes with coefficients b_l which are singular at $\rho=0$ for all l. The component R_2 contains singular modes with coefficients $a_N(\varepsilon>0), a_{-N-1}(\varepsilon<0)$ and $b_{l\neq N}(\varepsilon>0), b_{l\neq -N-1}(\varepsilon<0)$. The appearance of singular modes is usually forbidden by the normalization condition (28) for radial functions. However, for unbounded potentials, and for the AB potential (43) in particular, this condition is less restrictive. The normalization condition (28) requires that integrals of the type

$$I_\mu=\int_0^\infty J_\mu^2(p_\perp\rho)\rho\,\mathrm{d}\rho \tag{45}$$

are convergent at small ρ, which takes place at $\mu>-1$. For the positron solution $(\varepsilon>0)$ the normalization condition (45) for R_1 allows non-zero coefficients a_l, b_N and b_{N+1}, while this condition for R_2 allows non-zero coefficients a_l, b_N and b_{N-1}. For the Dirac equation both components R_1 and R_2 must satisfy the normalization condition (45). This means that only the coefficient b_N can be non-zero presenting one singular mode with $l=N$ in the positron solution of the Dirac equation (while two singular modes are allowed for scalar wave equations). For the electron solution $(\varepsilon<0)$ the normalization condition (45) for R_1 allows non-zero coefficients a_l, b_{-N} and b_{-N-1} while the condition for R_2 allows non-zero coefficients a_l, b_{-N-1} and b_{-N-2}. Therefore, only one singular mode with $l=-N-1$ can be present in the electron solution of the Dirac equation.

The problem of the singular modes is related to the fact that the Dirac operator as well as any Hamilton operator for charged particles, is not self-adjoint in the presence of the AB potential; this would cause many problems for the unitary evolution of quantum systems unless self-adjoint extensions of these Hamilton operators exist. The self-adjoint extension procedure applied to this problem gives results which can be obtained by direct calculation of the normalization integrals (for a detailed discussion see [3]). However, this procedure does not fix the extension parameters which determine the behaviour of the wavefunction at the origin. This situation is not satisfactory from the physical point of view. It can be overcome in turning to better defined models [9], in particular, to the model of a solenoid of finite radius with uniform magnetic field [4].

From (42) we obtain for $\rho_0 \to 0$

$$|l| - \beta + 2\beta \frac{A_1}{B_1} \frac{\Phi(A_1+1, B_1+1; \beta)}{\Phi(A_1, B_1; \beta)} \sim \nu \frac{\alpha - \xi_l}{\alpha + \xi_l} \tag{46}$$

where

$$\alpha = \left(\frac{p_\perp \rho_0}{2}\right)^{2\nu} \frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)} \tag{47}$$

and

$$\xi_l := \frac{b_l}{a_l} \sim \alpha \frac{\nu - |l| + \beta - 2\beta(A_1/B_1)(\Phi(A_1+1, B_1+1; \beta)/\Phi(A_1, B_1; \beta))}{\nu + |l| - \beta + 2\beta(A_1/B_1)(\Phi(A_1+1, B_1+1; \beta)/\Phi(A_1, B_1; \beta))}. \tag{48}$$

The asymptotic behaviour of ξ_l for $\rho_0 \to 0$ depends on the behaviour of the denominator in (48). Since $\nu > 0$, we have $b_l \sim \rho_0^{2\nu} \to 0$ for $\rho_0 \to 0$ unless the following equality is valid:

$$\lim_{\rho_0 \to \infty} D := \lim_{\rho_0 \to \infty} \left(\nu + |l| - \beta + 2\beta \frac{A_1}{B_1} \frac{\Phi(A_1+1, B_1+1; \beta)}{\Phi(A_1, B_1; \beta)} \right) = 0. \tag{49}$$

This can happen only if

$$\nu + |l| - \beta = 0 \qquad \text{and} \qquad \lim_{\rho_0 \to \infty} A_1 = |l| - \varepsilon l + 1 - \varepsilon = 0. \tag{50}$$

In this case the parameter ξ_l goes to zero,

$$\xi_l \sim \rho_0^{2(\nu-1)} \to 0 \qquad \text{at } \rho_0 \to \infty \tag{51}$$

unless

$$\nu = |l - \phi| < 1. \tag{52}$$

Equation (50) and (52) are fulfilled for the positron ($\varepsilon > 0$) singular mode $l = N$ only. This means that $a_N = 0, b_N \neq 0$, and that accordingly the first positron component with $l = N$ is singular and the second component is regular at $\rho = 0$:

$$R_1(\rho) = J_{-\delta}(p_\perp \rho) \qquad R_2(\rho) = J_{1-\delta}(p_\perp \rho). \tag{53}$$

In this case the interaction between the positron magnetic moment and the string magnetic field is attractive for the first (upper) component of the wavefunction and it is repulsive for the second (down) component. For the electron ($\varepsilon < 0$) solution all coefficients $b_l = 0$. This means that the singular electron mode $l = -N - 1$ has a regular first component and a singular second component:

$$R_1(\rho) = J_{1-\delta}(p_\perp \rho) \qquad R_2(\rho) = -J_{-\delta}(p_\perp \rho). \tag{54}$$

Introducing a new notation, we can rewrite the positron and electron solutions of the Dirac equation in the presence of the magnetic string in the following final forms which are valid both for regular and singular modes,

$$\begin{aligned} R_1(\rho) &= \mathrm{e}^{\mathrm{i}\frac{1}{2}\pi|l-\varepsilon N|} J_{\nu_1}(p_\perp \rho) \\ R_2(\rho) &= \mathrm{e}^{\mathrm{i}\frac{1}{2}\pi|l-\varepsilon N|} \epsilon_{l-\varepsilon N} J_{\nu_2}(p_\perp \rho) = -\mathrm{i}\, \mathrm{e}^{\mathrm{i}\frac{1}{2}\pi|l+1-\varepsilon N|} J_{\nu_2}(p_\perp \rho) \end{aligned} \tag{55}$$

with

$$\nu_1 := \epsilon_{l-\varepsilon N}(l-\phi) \qquad \nu_2 := \epsilon_{l-\varepsilon N}(l+1-\phi) \qquad \epsilon_{l-\varepsilon N} := \begin{cases} 1 & \text{if } l \geqslant \varepsilon N \\ -1 & \text{if } l < \varepsilon N. \end{cases} \tag{56}$$

These radial solutions satisfy the normalization conditions (28).

Redefinition of l. The matrix elements of the QED processes (bremsstrahlung of electrons and positrons, pair production and annihilation) in the presence of the AB potential contain integrals over the products $\overline{\Psi}(x)\gamma_\mu A_\mu(x)\Psi(x)$, $\overline{\Psi}_c(x)\gamma_\mu A_\mu(x)\Psi_c(x)$, $\overline{\Psi}(x)\gamma_\mu A_\mu(x)\Psi_c(x)$, $\overline{\Psi}_c(x)\gamma_\mu A_\mu(x)\Psi(x)$, correspondingly, where $\Psi(x)$ and $\Psi_c(x)$ are the electron and charge conjugate positron functions. The integer number N disappears from all these matrix elements after the redefinition of the angular quantum number l, $l \to l + \varepsilon N$. This means that the matrix elements of the QED processes are independent of the integer part of the magnetic flux in units of the magnetic quantum. One can foresee this fact beforehand since the Dirac equation (2) with $\beta = N + \delta$ can be transformed to the Dirac equation with $\beta = \delta$ by means of the gauge transformation

$$\Psi(x) \to \Psi'(x) = \exp(-\mathrm{i}\varepsilon N\varphi)\Psi(x)$$
$$eA_i(x) := \frac{\varepsilon\beta}{\rho} \to eA_i'(x) = eA_i(x) - \mathrm{i}\exp(\mathrm{i}\varepsilon N\varphi)\nabla_i \exp(-\mathrm{i}\varepsilon N\varphi) = \frac{\varepsilon(\beta - N)}{\rho}.$$

All observable quantities are conserved under this gauge transformation.

5. Scattering of scalar and Dirac particles by a solenoid of finite radius

The expressions obtained above present the partial positron and electron wavefunctions in terms of cylindrical modes. These states do not describe outgoing particles with definite linear momenta at infinity. In order to calculate the cross section of QED processes we need the *positron and electron scattering wavefunctions*. In external fields there exist two independent exact solutions of the Dirac equation, $\Psi^{(\pm)}(\boldsymbol{p}; x)$, which behave at large distances like a plane wave (propagating in the direction $\boldsymbol{p}$ given by $p_x = p_\perp \cos\varphi_p$, $p_y = p_\perp \sin\varphi_p$, p_z) plus an outgoing or ingoing cylindrical wave, correspondingly. Because of the damping of the cylindrical waves at large distances we may use these superpositions instead of plane waves. To evaluate correctly the matrix elements in the presence of external fields we have to take wavefunctions for ingoing (outgoing) particles which contain outgoing (ingoing) cylindrical waves. We turn first to a discussion of scalar particles.

5.1. Low-energy scattering of scalar particles

The wavefunction for scalar particles can be found as the eigenfunction of the complete set of commuting operators (5):

$$\Psi(x) = \frac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}E_p t + \mathrm{i}p_3 z}\psi(\rho,\varphi) \qquad \psi(\rho,\varphi) = R_l(\rho)\,\mathrm{e}^{\mathrm{i}l\varphi}. \tag{57}$$

Its radial modes $R_l(\rho)$ obey the radial Klein–Gordon equation,

$$R_l'' + \frac{1}{\rho}R_l' - \left(\frac{l}{\rho} - eA_\varphi\right)^2 R_l + p_\perp^2 R_l = 0 \tag{58}$$

which is similar to (26) but does not contain the term $e(A_\varphi' + A_\varphi/\rho)$ which describes the spin–magnetic field interaction.

For realistic models the internal radial solutions are some regular functions $c_l R_l^{\rm int}(\rho)$. For model (30) of a solenoid with constant magnetic field this is

$$R_l^{\rm int}(\rho) = c_l x^{\frac{1}{2}|l|}\,{\rm e}^{-\frac{1}{2}x}\Phi(A,B;x) \qquad x = \frac{\beta}{\rho_0^2}\rho^2$$
$$A_l = \frac{|l|-l+1}{2} - \frac{p_\perp^2\rho_0^2}{4\beta} \qquad B_l = |l|+1. \tag{59}$$

The external solutions are similar to the first line of (39). It is convenient to rewrite them as follows

$$R_l(\rho) = (a_l + b_l\,{\rm e}^{-{\rm i}\pi\nu})J_\nu(p_\perp\rho) + {\rm i}b_l \sin\pi\nu H_\nu^{(1)}(p_\perp\rho)$$
$$\nu = |l-\phi| \tag{60}$$

where $H_\nu^{(1)}(x)$ are the Hankel function. The corresponding scattering wavefunctions can be obtained by the superposition of these cylindrical modes.

Since the plane wave term of the scattering wavefunction at $\rho\to\infty$ is defined only by the external solution, the scattering wavefunction can be written as follows

$$\Psi(\boldsymbol{p};x) = \frac{1}{2\pi}\,{\rm e}^{-{\rm i}E_p t + {\rm i}p_3 z}\psi(\boldsymbol{p};\rho,\varphi) \tag{61}$$
$$\begin{aligned}\psi(\boldsymbol{p};\rho,\varphi) &= \sum_l \tilde{c}_{l-\varepsilon N}(\varphi_p)\,{\rm e}^{{\rm i}\frac{1}{2}\pi|l-\varepsilon N|}[J_\nu(p_\perp\rho) - \mu_{l-\varepsilon N}(\varphi_p)H_\nu^{(1)}(p_\perp\rho)]\,{\rm e}^{{\rm i}l\varphi}\\ &= {\rm e}^{{\rm i}\varepsilon N(\varphi-\varphi_p-\pi)}\sum_l {\rm e}^{{\rm i}\frac{1}{2}\pi|l|}\,{\rm e}^{{\rm i}\frac{1}{2}\pi(|l|-|l-\varepsilon\delta|)}\\ &\quad\times[J_{|l-\varepsilon\delta|}(p_\perp\rho) - \mu_l(\varphi_p)H^{(1)}_{|l-\varepsilon\delta|}(p_\perp\rho)]\,{\rm e}^{{\rm i}l(\varphi-\varphi_p)}\end{aligned}$$

with the coefficients

$$\tilde{c}_{l-\varepsilon N}(\varphi_p) := {\rm e}^{-{\rm i}\varepsilon N\pi - {\rm i}l\varphi_p}\,{\rm e}^{\pm{\rm i}\frac{1}{2}\pi(|l-\varepsilon N|-\nu)}. \tag{62}$$

The wavefunction (61) behaves asymptotically as follows

$$\psi(\boldsymbol{p};\rho,\varphi) \sim {\rm e}^{{\rm i}\varepsilon N(\varphi-\varphi_p-\pi)} \times \left[{\rm e}^{{\rm i}p_\perp\rho\cos(\varphi-\varphi_p)} + f(\varphi-\varphi_p)\frac{{\rm e}^{\pm{\rm i}p_\perp\rho}}{\sqrt{\rho}}\right] \tag{63}$$

where

$$f(\varphi-\varphi_p) = \frac{{\rm e}^{-{\rm i}\frac{1}{4}\pi}}{\sqrt{2\pi p_\perp}}\,{\rm e}^{-{\rm i}\varepsilon N(\varphi-\varphi_p)}\sum_l[{\rm e}^{{\rm i}\pi(|l-\varepsilon N|-\nu)}[1-2\mu_{l-\varepsilon N}(\varphi_p)]-1]\,{\rm e}^{{\rm i}l(\varphi-\varphi_p)} \tag{64}$$

is the scattering amplitude. We note that the distorting factor ${\rm e}^{{\rm i}\varepsilon N(\varphi-\varphi_p-\pi)}$ appears in (63) and (64) because the vector potential (30) decreases slowly at infinity.

The wavefunction (61) and the scattering amplitude (64) contain an arbitrary coefficient $\mu_l(\varphi_p)$ connected with the coefficients a_l and b_l of (60). It describes how the finite radius ρ_0 of the solenoid influences the scattering cross section and can be found from the corresponding matching condition:

$$\mu_{l-\varepsilon N}(\varphi_p) = \frac{J_\nu(p_\perp\rho_0)}{H_\nu^{(1)}(p_\perp\rho_0)}\,\frac{(\ln J_\nu(p_\perp\rho_0))' - (\ln R_l^{\rm int}(p_\perp\rho_0))'}{(\ln H_\nu^{(1)}(p_\perp\rho_0))' - (\ln R_l^{\rm int}(p_\perp\rho_0))'}. \tag{65}$$

The partial wave decomposition is the effective method for the investigation of low-energy scattering. In this case the de Broglie wavelength of scattered particles is large compared with the size of the target (ρ_0), and the lower angular momenta contributes mainly to the scattering cross section. One can see from (65) that $\mu_l\to 0$ at $\rho_0\to 0$. For any realistic model, including the model (30), the internal solution is regular at $\rho_0 = 0$. Then we have

$$\frac{R_l'^{\mathrm{int}}(p_\perp\rho_0)}{R_l^{\mathrm{int}}(p_\perp\rho_0)} \sim \frac{1}{p_\perp\rho_0} \qquad L_{l-\varepsilon N} = \lim_{\rho_0\to 0} \frac{(\ln J_\nu(p_\perp\rho_0))' - (\ln R_l^{\mathrm{int}}(p_\perp\rho_0))'}{(\ln H_\nu^{(1)}(p_\perp\rho_0))' - (\ln R_l^{\mathrm{int}}(p_\perp\rho_0))'} \sim \text{constant} \tag{66}$$

and the second term in (61) disappears completely at $\rho_0 \to 0$,

$$\mu_{l-\varepsilon N}(\varphi_p) \sim \mathrm{i}\sin\pi\nu \frac{\Gamma(-\nu+1)}{\Gamma(\nu+1)} L_{l-\varepsilon N} \left(\frac{p_\perp\rho_0}{2}\right)^{2\nu} \to 0 \tag{67}$$

with

$$L_{l-\varepsilon N} = \frac{\nu - |l| + \beta - 2\beta\eta_l}{-\nu - |l| + \beta - 2\beta\eta_l} \qquad \eta_l = \frac{\Phi'(A_1, B_1; \beta)}{\Phi(A_1, B_1; \beta)}. \tag{68}$$

Accordingly, we have obtained the following result: at low energies, $p_\perp\rho_0 \to 0$, the scattering wavefunction (61) coincides with the AB wavefunction which behaves asymptotically as follows

$$\psi(\boldsymbol{p}; \rho, \varphi) \sim \mathrm{e}^{\mathrm{i}\varepsilon N(\varphi-\varphi_p-\pi)} \left[\mathrm{e}^{\mathrm{i}\varepsilon\delta(\varphi-\varphi_p-\pi)} \, \mathrm{e}^{\mathrm{i}p_\perp\rho\cos(\varphi-\varphi_p)} + f_{\mathrm{AB}}(\varphi-\varphi_p) \frac{\mathrm{e}^{\mathrm{i}p_\perp\rho}}{\sqrt{\rho}} \right] \tag{69}$$

in the whole space outside a narrow region around the forward direction. The corresponding AB scattering amplitude reads

$$f_{\mathrm{AB}}(\varphi-\varphi_p) = -\varepsilon \frac{1}{\sqrt{2\pi p_\perp}} \, \mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi} \, \mathrm{e}^{\mathrm{i}\varepsilon\frac{1}{2}(\varphi-\varphi_p)} \frac{\sin\pi\delta}{\sin\frac{1}{2}(\varphi-\varphi_p)} \tag{70}$$

and the differential cross section of the AB scattering [1] is equal to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = \frac{p_\perp}{p} |f_{\mathrm{AB}}(\varphi-\varphi_p)|^2 = \frac{1}{2\pi p} \frac{\sin^2\pi\delta}{\sin^2\frac{1}{2}(\varphi-\varphi_p)}. \tag{71}$$

To find the first correction to the AB scattering amplitude at small ρ_0 one needs to take a minimal value of ν. It is δ at $0 < \delta < 1/2$, $l = \varepsilon N$ and $1-\delta$ at $1/2 < \delta < 1$, $l = \varepsilon(N+1)$. We obtain

$$\begin{aligned} \mu_0(\varphi_p) &= \mathrm{i}\sin\pi\delta \frac{\Gamma(1-\delta)}{\Gamma(1+\delta)} L_0 \left(\frac{p_\perp\rho_0}{2}\right)^{2\delta} \\ \mu_\varepsilon(\varphi_p) &= \mathrm{i}\sin\pi\delta \frac{\Gamma(\delta)}{\Gamma(2-\delta)} L_\varepsilon \left(\frac{p_\perp\rho_0}{2}\right)^{2(1-\delta)} \end{aligned} \tag{72}$$

with

$$L_0 = 1 - \frac{\delta}{(N+\delta)\eta_{\varepsilon N}} \qquad L_\varepsilon = \frac{1}{1 + (1-\delta)/(N+\delta)\eta_{\varepsilon(N+1)}^{-1}}. \tag{73}$$

Then the scattering wavefunction reads as follows

$$\begin{aligned} \psi(\boldsymbol{p}; \rho, \varphi) &= \psi_{\mathrm{AB}}(\boldsymbol{p}; \rho, \varphi) + \delta\psi(\boldsymbol{p}; \rho, \varphi) \\ \delta\psi(\boldsymbol{p}; \rho, \varphi) &= -\mu_0(\varphi_p)\, \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi\delta} H_\delta^{(1)}(p_\perp\rho_0)\, \mathrm{e}^{\mathrm{i}\varepsilon N(\varphi-\varphi_p-\pi)} - \mu_\varepsilon(\varphi_p)\, \mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi(1-\delta)} H_{1-\delta}^{(1)}(p_\perp\rho_0) \\ &\quad \times \mathrm{e}^{\mathrm{i}\varepsilon(N+1)(\varphi-\varphi_p-\pi)} \end{aligned} \tag{74}$$

where $\psi_{\mathrm{AB}}(\boldsymbol{p};\rho,\varphi)$ is the AB scattering wavefunction with the asymptotic behaviour (69). Now the additional term changes the modes with $l = \varepsilon N$, $\varepsilon(N+1)$ of the scattering amplitude $f_{\mathrm{AB}}(\varphi-\varphi_p)$ given by (70)

$$f(\varphi-\varphi_p) = f_{\mathrm{AB}}(\varphi-\varphi_p) + \delta f(\varphi-\varphi_p)$$
$$\delta f(\varphi-\varphi_p) = -\frac{2}{\sqrt{2\pi p_\perp}}\,\mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi}(\mu_0\,\mathrm{e}^{-\mathrm{i}\pi\delta} + \mu_\varepsilon\,\mathrm{e}^{-\mathrm{i}\pi(1-\delta)}\,\mathrm{e}^{\mathrm{i}\varepsilon(\varphi-\varphi_p-\pi)}). \tag{75}$$

In this approximation the scattering cross section is equal to:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = \frac{p_\perp}{p}|f(\varphi-\varphi_p)|^2 = \frac{1}{2\pi p}\frac{\sin^2\pi\delta}{\sin^2\frac{1}{2}(\varphi-\varphi_p)}(1+\Delta)$$
$$\Delta = 4\sin\frac{\varphi-\varphi_p}{2}\sin\left(\frac{\varphi-\varphi_p}{2}+\pi\varepsilon\delta\right)$$
$$\times\left[\frac{\Gamma(1-\delta)}{\Gamma(1+\delta)}L_0\left(\frac{p_\perp\rho_0}{2}\right)^{2\delta} + \frac{\Gamma(\delta)}{\Gamma(2-\delta)}L_1\left(\frac{p_\perp\rho_0}{2}\right)^{2(1-\delta)}\right] \tag{76}$$

whereby $\Delta = 0$ for $\rho_0 = 0$.

5.2. *High-energy scattering of scalar particles*

The partial wave method becomes ineffective when applied to the scattering of high-energy particles. The smaller the de Broglie wavelength of scattered particles, the higher angular momenta contribute to the scattering cross section. At $p_\perp\rho_0 \to \infty$ this corresponds to $l \to \infty$. In this case the scattering process can be expected to be quasiclassical and the WKB method [11] can be used to obtain the scattering cross section.

The quasiclassical radial wavefunction for the model (30) is equal to

$$\psi^{\mathrm{WKB}} = \text{constant }F_l^{-\frac{1}{4}}(\rho)\exp\left[\mathrm{i}\int\mathrm{d}\rho'\sqrt{F_l(\rho')}\right] \qquad F_l(\rho) = p_\perp^2 - (l - e\rho A_\varphi)^2. \tag{77}$$

Then the phase shift in the WKB approximation is given by

$$\delta_l^{\mathrm{WKB}} = \frac{1}{2}\pi l - \int_{\rho_{\mathrm{min}}}^{\infty}\mathrm{d}\rho'[\sqrt{F_l(\rho)} - p_\perp] - p_\perp\rho_{\mathrm{min}} \tag{78}$$

where we have introduced the radius of the orbit in the uniform magnetic field $R = p_\perp/eB$. ρ_{min} is the classical turning point of the motion inside the solenoid where the radial kinetic energy $F_l(\rho)$ goes to zero. $F_l(\rho)$ contains ρ_0 via A_φ, and in the zero radius limit (78) agrees with the corresponding expression for AB phase shift.

The classical deflection angle for charged particle moving in the magnetic field of the solenoid of finite radius (30) can be obtained from the corresponding radial Hamilton–Jacobi equation. The WKB method gives the same result. Differentiating the phase shift (78) with respect to l we obtain the classical deflection angle

$$\varphi = 2\frac{\mathrm{d}}{\mathrm{d}l}\delta_l^{\mathrm{WKB}} = \pi - 2\int_{\rho_{\mathrm{min}}}^{\infty}\frac{\mathrm{d}\rho}{\rho}\frac{(l-(p_\perp\rho^2/2R))}{\sqrt{p_\perp^2\rho^2 - (l-(p_\perp\rho^2/2R))^2}}$$
$$-2\int_{\rho_{\mathrm{min}}}^{\infty}\frac{\mathrm{d}\rho}{\rho}\frac{(l-(p_\perp\rho_0^2/2R))}{\sqrt{p_\perp^2\rho^2 - (l-(p_\perp\rho_0^2/2R))^2}}. \tag{79}$$

Introducing the impact parameter

$$a := \frac{l}{p_\perp} - \frac{\rho_0^2}{2R} \tag{80}$$

and calculating the integral (79) we obtain

$$\tan\frac{\varphi}{2} = \frac{\sqrt{\rho_0^2 - a^2}}{R + a}. \tag{81}$$

One can see from (81) that one or two values of the deflection angle correspond to one and the same value of the impact parameter a. In this case classically unobtainable interference effects arise in the framework of the WKB approximation. Since the difference between the two impact parameters that lead to the same deflection angle is of macroscopic size in comparing with the particle wavelength these interference effects are unobservable and can be neglected.

Then we obtain the fact that the differential scattering cross section of scalar particles by the magnetic tube of the finite radius ρ_0 in the WKB approximation is equal to the classical cross section

$$\frac{\mathrm{d}\sigma(\varphi)}{\mathrm{d}\varphi} = \left|\frac{\mathrm{d}a}{\mathrm{d}\varphi}\right| = \begin{cases} \dfrac{1}{2}\sin\dfrac{\varphi}{2}\dfrac{(\sqrt{\rho_0^2 - R^2\sin^2(\varphi/2)} + R\cos^2(\varphi/2))^2}{\sqrt{\rho_0^2 - R^2\sin^2(\varphi/2)}} & \text{at } R < \rho_0 \\ \sin\dfrac{\varphi}{2}\dfrac{\rho_0^2 - R^2\sin^2(\varphi/2) + R^2\cos^2(\varphi/2)}{\sqrt{\rho_0^2 - R^2\sin^2(\varphi/2)}} & \text{at } R > \rho_0. \end{cases} \tag{82}$$

Integrating over φ, we find that the total scattering cross section of high-energy incident particles is equal to $\sigma = 2\rho_0$. This means that the scattering cross section decreases with the energy of incident particles approaching its geometrical value.

5.3. Low-energy scattering of Dirac particles

In this section we consider the scattering of Dirac particles of low energies by the solenoid of finite radius ρ_0. As the initial approximation at $\rho_0 \to 0$ we prefer to use the radial function in the form of (55) and (56) instead of (39). With this choice we avoid the problem connected with the specific behaviour of the parameter ξ_l of (48) at $\rho_0 \to 0$ with $l = N$ for the positron states and $l = -N - 1$ for the electron states.

Then, for the solenoid of finite radius, the external positron and electron radial solutions read as follows

$$\begin{aligned} R_1(\rho) &= \mathrm{e}^{\mathrm{i}\frac{1}{2}\pi|l-\varepsilon N|}[J_{\nu_1}(p_\perp\rho) - \mu_{l-\varepsilon N}H^{(1)}_{\nu_1}(p_\perp\rho)] \\ R_2(\rho) &= -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\pi|l+1-\varepsilon N|}[J_{\nu_2}(p_\perp\rho) - \mu_{l-\varepsilon N}H^{(1)}_{\nu_2}(p_\perp\rho)] \end{aligned} \tag{83}$$

with the orders of the Bessel functions given by (56). The coefficients μ_l are defined by the corresponding matching conditions and decrease rapidly for $\rho_0 \to 0$. With these radial functions the cylindrical modes for the Dirac equation (2) are given by (6) and (7) with $R_3 = R_1$, $R_4 = R_2$ and the spin coefficients (15), (16) or (20) and (21) correspondingly.

The scattering wavefunctions for the Dirac equation (2),

$$\psi_{\mathrm{D}}(\boldsymbol{p};\rho,\varphi) = \begin{pmatrix} u(\boldsymbol{p};\rho,\varphi) \\ v(\boldsymbol{p};\rho,\varphi) \end{pmatrix} \tag{84}$$

are defined by the series

$$u(\boldsymbol{p};\rho,\varphi) = \sum_l c_{l-\varepsilon N}\begin{pmatrix} C_1 R_1(\rho)\,\mathrm{e}^{\mathrm{i}l\varphi} \\ C_2 R_2(\rho)\,\mathrm{e}^{\mathrm{i}(l+1)\varphi} \end{pmatrix} = \begin{pmatrix} C_1\psi_1(\boldsymbol{p};\rho,\varphi) \\ -\mathrm{i}C_2\,\mathrm{e}^{\mathrm{i}\varphi_p}\psi_2(\boldsymbol{p};\rho,\varphi) \end{pmatrix} \tag{85}$$

$$v(\boldsymbol{p};\rho,\varphi)=\sum_l c_{l-\varepsilon N}\begin{pmatrix} C_3R_1(\rho)\,\mathrm{e}^{\mathrm{i}l\varphi} \\ C_4R_2(\rho)\,\mathrm{e}^{\mathrm{i}(l+1)\varphi}\end{pmatrix}=\begin{pmatrix} C_3\psi_1(\boldsymbol{p};\rho,\varphi) \\ -\mathrm{i}C_4\,\mathrm{e}^{\mathrm{i}\varphi_p}\psi_2(\boldsymbol{p};\rho,\varphi)\end{pmatrix} \tag{86}$$

with the coefficients

$$c_{l-\varepsilon N}(\varphi_p)=\mathrm{e}^{-\mathrm{i}\varepsilon N\pi-il\varphi_p}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\pi\varepsilon\epsilon_{l-\varepsilon N}\,\delta}. \tag{87}$$

These coefficients differ from the coefficients $\tilde{c}_{l-\varepsilon N}(\varphi_p)$ (62) of the scattering wavefunction for the scalar particles (61) only at $l=N$ for the positron mode and at $l=-N-1$ for the electron mode.

Using equation (83) we find the components of the Dirac scattering wavefunction for the positron solution

$$\begin{aligned}\psi_1(\boldsymbol{p};\rho,\varphi)&=\psi(\boldsymbol{p};\rho,\varphi)+\mathrm{e}^{\mathrm{i}N(\varphi-\varphi_p-\pi)}\,\mathrm{i}\sin\pi\delta\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\pi\delta}H^{(1)}_\delta(p_\perp\rho)+\delta\psi_1(\boldsymbol{p};\rho,\varphi)\\ \psi_2(\boldsymbol{p};\rho,\varphi)&=\psi(\boldsymbol{p};\rho,\varphi)+\delta\psi_2(\boldsymbol{p};\rho,\varphi)\end{aligned} \tag{88}$$

and for the electron solution

$$\begin{aligned}\psi_1(\boldsymbol{p};\rho,\varphi)&=\psi(\boldsymbol{p};\rho,\varphi)+\delta\psi_1(\boldsymbol{p};\rho,\varphi)\\ \psi_2(\boldsymbol{p};\rho,\varphi)&=\psi(\boldsymbol{p};\rho,\varphi)+\mathrm{e}^{-\mathrm{i}N(\varphi-\varphi_p-\pi)}\,\mathrm{i}\sin\pi\delta\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\pi\delta}H^{(1)}_\delta(p_\perp\rho)+\delta\psi_2(\boldsymbol{p};\rho,\varphi).\end{aligned} \tag{89}$$

Here the function $\psi(\boldsymbol{p};\rho,\varphi)$ is the scattering wavefunction (61) for scalar particles, and the terms of the first approximation are equal to

$$\begin{aligned}\delta\psi_1(\boldsymbol{p};\rho,\varphi)&=-\mathrm{e}^{\mathrm{i}N(\varphi-\varphi_p-\pi)}\sum_l \mathrm{e}^{\mathrm{i}\frac{1}{2}\pi|l|}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\pi\epsilon_l\,\delta}\mu_l H^{(1)}_{\epsilon_l(l-\delta)}(p_\perp\rho)\,\mathrm{e}^{\mathrm{i}l(\varphi-\varphi_p)}\\ \delta\psi_2(\boldsymbol{p};\rho,\varphi)&=-\mathrm{e}^{\mathrm{i}N(\varphi-\varphi_p-\pi)}\sum_l \mathrm{e}^{\mathrm{i}\frac{1}{2}\pi|l+1|}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\pi(|l+1|-|l+1-\delta|)}\mu_l\,H^{(1)}_{|l+1-\delta|}(p_\perp\rho)\,\mathrm{e}^{\mathrm{i}(l+1)(\varphi-\varphi_p)}\end{aligned} \tag{90}$$

for the positron solution, and

$$\begin{aligned}\delta\psi_1(\boldsymbol{p};\rho,\varphi)&=-\mathrm{e}^{-\mathrm{i}N(\varphi-\varphi_p-\pi)}\sum_l \mathrm{e}^{\mathrm{i}\frac{1}{2}\pi|l|}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}\pi(|l|-|l+\delta|)}\mu_l H^{(1)}_{|l+\delta|}(p_\perp\rho)\,\mathrm{e}^{\mathrm{i}l(\varphi-\varphi_p)}\\ \delta\psi_2(\boldsymbol{p};\rho,\varphi)&=-\mathrm{e}^{-\mathrm{i}N(\varphi-\varphi_p-\pi)}\sum_l \mathrm{e}^{\mathrm{i}\frac{1}{2}\pi|l+1|}\,\mathrm{e}^{-\mathrm{i}\frac{1}{2}\pi\epsilon_l\delta}\mu_l H^{(1)}_{\epsilon_l(l+1+\delta)}(p_\perp\rho)\,\mathrm{e}^{\mathrm{i}(l+1)(\varphi-\varphi_p)}\end{aligned} \tag{91}$$

for the electron solution. These terms decrease rapidly for $\rho_0\to 0$. However, (88) and (89) contain the additional terms which appear in the AB scattering wavefunctions for the Dirac particles for $\rho_0\to 0$ and which are absent for scalar particles. They arise due to a distortion of the positron radial mode with $l=N$ and the electron radial mode with $l=-1-N$ caused by the interaction the magnetic momenta of the Dirac particles with the string magnetic fields. This interaction does not influence other modes which go to zero on the string.

At $p_\perp\rho\to\infty$, and taking into account (69), we obtain the asymptotic behaviour of the Dirac scattering wavefunction (84)–(89),

$$\begin{aligned}\psi(\boldsymbol{p};\rho,\varphi)\sim{}&\mathrm{e}^{\mathrm{i}\varepsilon N(\varphi-\varphi_p-\pi)}\left[\mathrm{e}^{\mathrm{i}\varepsilon\delta(\varphi-\varphi_p-\pi)}\,\mathrm{e}^{\mathrm{i}p_\perp\rho\cos(\varphi-\varphi_p)}+F(\varphi-\varphi_p)G(\varphi-\varphi_p)\frac{\mathrm{e}^{\mathrm{i}p_\perp\rho}}{\sqrt{\rho}}\right]\\ &\times\begin{pmatrix}u(\boldsymbol{p})\\ v(\boldsymbol{p})\end{pmatrix}\end{aligned} \tag{92}$$

where the scattering amplitude matrix $F(\varphi-\varphi_p)G(\varphi-\varphi_p)$ is defined by the expressions

$$\begin{aligned}F(\varphi-\varphi_p)&=f_{\mathrm{AB}}(\varphi-\varphi_p)\,\mathrm{e}^{-\mathrm{i}\varepsilon\frac{1}{2}(\varphi-\varphi_p)}+\delta f(\varphi-\varphi_p)\,\mathrm{e}^{\mathrm{i}\frac{1}{2}(\varphi-\varphi_p)}\\ \delta f(\varphi-\varphi_p)&=\sqrt{\frac{2}{\pi p_\perp}}\,\mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi}\sum_l \mathrm{e}^{\mathrm{i}\pi\varepsilon\epsilon_l\delta}\mu_l\,\mathrm{e}^{\mathrm{i}l(\varphi-\varphi_p)}\end{aligned} \tag{93}$$

and

$$G(\varphi-\varphi_p)=\begin{pmatrix} D(\varphi-\varphi_p) & 0 \\ 0 & D(\varphi-\varphi_p)\end{pmatrix} \tag{94}$$

with

$$D(\varphi-\varphi_p)=\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\frac{1}{2}(\varphi-\varphi_p)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\frac{1}{2}(\varphi-\varphi_p)}\end{pmatrix}=\cos\frac{1}{2}(\varphi-\varphi_p)-\mathrm{i}\sigma_3\sin\frac{1}{2}(\varphi-\varphi_p). \tag{95}$$

The coefficients μ_l in (93) are defined by the matching conditions and go to zero at $\rho_0\to 0$.

Here the plane wave bispinor

$$\psi(\boldsymbol{p})=\begin{pmatrix} u(\boldsymbol{p}) \\ v(\boldsymbol{p})\end{pmatrix} \qquad u(\boldsymbol{p})=\begin{pmatrix} C_1 \\ -\mathrm{i}C_2\,\mathrm{e}^{\mathrm{i}\varphi_p}\end{pmatrix} \qquad v(\boldsymbol{p})=\begin{pmatrix} C_3 \\ -\mathrm{i}C_4\,\mathrm{e}^{\mathrm{i}\varphi_p}\end{pmatrix} \tag{96}$$

describes the ingoing particle with the linear momentum $\boldsymbol{p}$.

The differential cross section of the scattering for the Dirac particles is equal to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}=\frac{\boldsymbol{\rho}\cdot\boldsymbol{j}}{j_p}=|F(\varphi-\varphi_p)|^2\frac{\psi^\dagger(\boldsymbol{p})G^\dagger(\varphi-\varphi_p)\alpha_\rho(\varphi)G(\varphi-\varphi_p)\psi(\boldsymbol{p})}{\psi^\dagger(\boldsymbol{p})\alpha_{\boldsymbol{p}}(\varphi)\psi(\boldsymbol{p})} \tag{97}$$

where $\boldsymbol{\rho}\cdot\boldsymbol{j}$ and j_p are flux densities of the scattered cylindrical and the incident plane waves.

Since $G^\dagger(\varphi-\varphi_p)\alpha_\rho(\varphi)G(\varphi-\varphi_p)=\alpha_\rho(\varphi_p)$, and the relations

$$\psi^\dagger(\boldsymbol{p})\alpha_\rho(\varphi_p)\psi(\boldsymbol{p})=\frac{p_\perp}{E_p}$$

$$\psi^\dagger(\boldsymbol{p})\alpha_{\boldsymbol{p}}(\varphi_p)\psi(\boldsymbol{p})=\psi^\dagger(\boldsymbol{p})\left(\frac{p_\perp}{p}\alpha_\rho(\varphi_p)+\frac{p_3}{p}\alpha_z\right)\psi(\boldsymbol{p})=\frac{p}{E_p} \tag{98}$$

are valid both for the helicity and the transverse states, we find that the differential cross section of the scattering of positrons and electrons by the solenoid of the finite radius

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}=\frac{p_\perp}{p}|F(\varphi-\varphi_p)|^2 \tag{99}$$

is independent of the spin polarization quantum number s. It coincides with the differential cross section for scalar particles. This means that spin effects do not appear in the scattering of spin particles in given helicity or transverse polarization states which are conserved in this process.

For particles polarized along an arbitrary vector $\boldsymbol{n}$ and described by the density matrix

$$\mathcal{R}(\boldsymbol{n})=\tfrac{1}{2}(1+\boldsymbol{\sigma}\cdot\boldsymbol{n}) \tag{100}$$

the differential cross section can be calculated by the formula

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}=\frac{p_\perp}{p}|F(\varphi-\varphi_p)|^2\,\mathrm{Tr}\,\mathcal{R}(\boldsymbol{n}')D(\varphi-\varphi_p)\mathcal{R}(\boldsymbol{n})D^\dagger(\varphi-\varphi_p) \tag{101}$$

where $\boldsymbol{n}'$ is the direction of the polarization vector after the scattering.

Calculating the trace on the right-hand side of (101) we find the differential cross section for particle beams of the given polarization

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}=\frac{p_\perp}{p}|F(\varphi-\varphi_p)|^2\cdot\frac{1}{2}\{&(1+\boldsymbol{n}\boldsymbol{n}')+(\boldsymbol{n}\boldsymbol{n}'-n_3n_3')\cos(\varphi-\varphi_p)\\ &+[\boldsymbol{n}\times\boldsymbol{n}']_3\sin(\varphi-\varphi_p)\}.\end{aligned} \tag{102}$$

For the case of the same polarizations of the scattered and the incident beams, i.e. at $\boldsymbol{n}=\boldsymbol{n}'$, we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi}=\frac{p_\perp}{p}|F(\varphi-\varphi_p)|^2\left(\cos^2\frac{\varphi-\varphi_p}{2}+n_3^2\sin^2\frac{\varphi-\varphi_p}{2}\right). \tag{103}$$

One can see that the spin effect occurs for polarized particles beams.

6. Conclusion

We have studied in detail the Dirac equation in cylindrical magnetic fields of a fixed direction and found eigenfunctions of a complete set of commuting operators which consists of the Dirac operator itself, the z-components of linear and total angular momenta, and of one of two possible spin polarization operators. The spin structure of the solutions is independent of the radial distribution of the magnetic field and is fixed by one of the spin polarization operators. The magnetic field influences only the radial modes. We have solved explicitly the radial equations for the model where the magnetic field is distributed uniformly inside a solenoid of finite radius. From this realistic model we have obtained correct solutions of the Dirac equation for the AB potential (magnetic string) in the zero radius limit. Also we have considered carefully the scattering of scalar and Dirac particles by a magnetic tube of finite radius. For low-energy particles, the scattering cross section coincides with the AB scattering cross section. In this case the de Broglie wavelength of particles is bigger compared with the tube radius and the influence of enclosed magnetic flux leads to the large cross section. This mechanism of interaction of quantum particles with the string magnetic field is more effective than the local interaction with the magnetic field. We have calculated the first-order corrections connected with the finite radius of the tube. When the energy of incident particles increases, the influence of the enclosed magnetic flux weakens and the cross section decreases. At high energies we obtained, therefore, the classical result for the scattering cross section.

The exact solutions derived above will also serve as a basis for future calculations. We plan to study how the finite size of the solenoid influences the differential cross sections of the bremsstrahlung emitted by an electron and the pair production by a single photon at different energies of the incoming particles.

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